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Guidance Strategy for Radially Accelerated Trajectories

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A guidance scheme is proposed for orbital motion under continuous outward radial acceleration that is inversely proportional to the square of the radial distance from the sun. Such an acceleration regime would be realized under the minimagnetospheric plasma propulsion. The maximum attainable radial distance of the outbound trajectory is investigated, and a guidance scheme for achieving this target maximum distance is established under radial acceleration disturbances. The scheme not only provides a control law for continuous radial acceleration but also yields the amount and timing of impulsive maneuvers required to satisfy the guidance requirement at the terminal point.

Nomenclature

a	=	nondimensional semimajor axis
a_r	=	spacecraft acceleration component in the radial direction
a_θ	=	spacecraft acceleration component in the circumferential direction
$E[\ ]$	=	expectation operator
g	=	magnitude of inward gravitational attraction of the sun
i	=	subscript indicating radial distance node for correction maneuvers
$\mathrm{MAX}(\Delta\rho_{\max})$	=	specified allowable maximum radial distance deviation
n	=	number of correction nodes
P	=	nondimensional orbital period
p	=	positional navigation error in radial direction
r	=	distance from the center of attraction (polar coordinate)
t	=	time
v	=	radial velocity navigation error
α	=	sensitivity of continuous acceleration error to the maximum attainable radial distance
β	=	sensitivity of radial velocity increment (or radial velocity navigation error) to the maximum attainable radial distance
γ	=	sensitivity of positional navigation error to the maximum attainable radial distance
ΔV	=	impulsive maneuver velocity increment
$\Delta\varepsilon$	=	continuous acceleration control margin
$\delta(\Delta V)$	=	impulsive maneuver mechanization error
$\delta(\Delta\varepsilon)$	=	continuous acceleration mechanization error
ε	=	nondimensional acceleration
θ	=	angular position (polar coordinate)
λ	=	sensitivity of angular velocity navigation error to the maximum attainable radial distance
μ_{SUN}	=	gravity constant of the sun
ρ	=	nondimensional radial distance
ρ_{MAX}	=	maximum attainable radial distance
τ	=	nondimensional time
ψ	=	angular velocity navigation error

Subscripts

command	=	guidance command
predicted	=	predicted maximum radial distance
target	=	target maximum radial distance
total	=	total amount
0	=	quantities at the starting instant $t = 0$

Introduction

A GUIDANCE scheme for orbital motion under continuous outward radial acceleration is investigated. Although the classical problem of spacecraft trajectory under continuous radial acceleration has been investigated by many researchers in the past,[1–7] a guidance scheme has yet to be established. An analytical solution is available when the radial acceleration is applied to a spacecraft under the influence of the gravity field of the central body. Tsien,[1] Irving,[2] and Battin[3] showed that there exists a critical value for the constant radial acceleration above which the spacecraft will achieve escape conditions. When the acceleration is below the critical value, the spacecraft attains a maximum distance along an outbound trajectory. If the radial acceleration is maintained beyond the maximum distance, the spacecraft spirals back to the initial radius along an inbound trajectory. Boltz[4] treated the radial acceleration case for acceleration inversely proportional to the square of the distance from the central body using the equation of motion described by the velocity and flight-path angle. Prussing and Coverstone-Carroll[5] provided a nonlinear radial spring interpretation for the constant radial acceleration problem based on an energy-well approach. Akella[6] focused on the constant radial acceleration case and showed that although the time intervals for the outbound and inbound trajectories are identical the trajectories themselves are very different. Broucke and Akella[7] described the general types of solutions for the continuous constant outward radial acceleration problem, using numerical integrations and concepts such as the theory of periodic orbits and Poincare's characteristic exponents, rather than the closed-form analytical solution for flight time in terms of elliptic integrals.

The present treatment considers a mission to the outer planets of the solar system, where the spacecraft is propelled by continuous thrust in the outward radial direction. A guidance scheme for reaching the target planet is established based on a fundamental assumption that the acceleration is inversely proportional to the square of the radial distance from the central body. The propulsive force considered is minimagnetospheric plasma propulsion (M2P2), a technology proposed by Winglee et al.[8] Although the full proof of the technology has yet to be reported, M2P2 represents a promising propulsion technology for accelerating the spacecraft in the outward radial direction. It produces force by the interaction between the solar wind and an artificial, plasma-inflated magnetic field around the spacecraft. The acceleration by M2P2 is nearly constant if the power provided by the solar panel is constant, but in reality the available power will be approximately inversely proportional to the square

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of the radial distance from the sun, which in turn yields a radial acceleration that is inversely proportional to the square of the radial distance.

Equations of Motion

The problem considered is the motion of the spacecraft under the influence of the thrust of the spacecraft and the gravitational attraction of the sun. The acceleration caused by the thrust of the spacecraft is confined to the outward radial direction. Therefore, the trajectory remains in a plane and can be described by two-degrees-of-freedom equations of motion. Let the position of the spacecraft at any time instant t be given by the polar coordinates r and θ, where r is the distance from the sun. The equations of motion for the spacecraft are then given by

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = a_r \tag{1}$$

$$\frac{\mathrm{d}(r^2\dot{\theta})}{\mathrm{d}t} = r a_\theta \tag{2}$$

Because the acceleration caused by the thrust is in the outward radial direction and inversely proportional to the square of the radial distance from the sun, the components of the acceleration become

$$a_r = -(g_0 - a_0)\left(r_0^2 / r^2\right) \tag{3}$$

$$a_\theta = 0 \tag{4}$$

Then, let

$$\rho = r/r_0, \qquad \tau = \sqrt{g_0/r_0}\, t, \qquad \varepsilon = a_0/g_0 \tag{5}$$

The spacecraft is initially located in an Earth orbit, and the sun–Earth distance r_0 is normalized to 1. The gravitational acceleration at the starting instance g_0 is 0.006 m/s^2 given by μ_{SUN}/r_0^2, where $\mu_{\mathrm{SUN}} = 1.32712438 \times 10^{20}$ m^3/s^2 and $r_0 = 1.49597870 \times 10^{11}$ m. Equations (1) and (2) can be rewritten in nondimensional form as

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2} = \rho\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 - \frac{1}{\rho^2} + \varepsilon\frac{1}{\rho^2} \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\rho^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) = 0 \tag{7}$$

The initial conditions are as follows:

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right)_0 = 0 \tag{8}$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)_0 = 1 \tag{9}$$

According to Eq. (15) of Ref. 4, when the nondimensional acceleration ε is larger than 0.5 the spacecraft attains a parabolic trajectory and escapes, whereas under the condition $0 < \varepsilon < 0.5$ the maximum attainable distance $\rho_{\max}$ is given by

$$\rho_{\max} = 1/(1 - 2\varepsilon) \tag{10}$$

This corresponds to the apoapsis distance in two-body dynamics with an equivalent gravity constant of magnitude $(g_0 - a_0)r_0^2 = g_0 r_0^2(1-\varepsilon)$ according to Eq. (3). Therefore, the nondimensional semimajor axis can be derived using the fact that the nondimensional periapsis and apoapsis distances are 1 and $1/(1-2\varepsilon)$ respectively, as follows:

$$a = (1-\varepsilon)/(1-2\varepsilon) \tag{11}$$

The nondimensional orbital period P is then expressed as

$$P = 2\pi\sqrt{(1-\varepsilon)^2/(1-2\varepsilon)^3} \tag{12}$$

Guidance Scheme for Radially Accelerated Trajectories

The investigation of the outbound trajectory pays attention to the maximum distance from the sun, that is, the location of the target planet. The objective of the guidance scheme would be to achieve the target maximum distance within the specified error allowance given navigational and acceleration error. A guidance scheme for achieving the target maximum distance is discussed here using an optimal spacing rule. The optimal spacing rule for ballistic cases was originally proposed by Breakwell[9] and Lawden and Long[10] and was later extended to cases of low thrust by Kawaguchi and Matsuo.[11] Both of these previous papers assume a sensitivity matrix of navigational error and impulsive maneuver for the terminal miss distance, which is proportional to the flight time to the terminal point. The guidance scheme proposed in this paper is further extended to the case of continuous radial acceleration by taking two-body dynamics into account, which yields a sensitivity matrix for the maximum radial distance that is a function of the radial distance. The proposed guidance scheme not only provides a control law for continuous radial acceleration, but also yields the required margin for continuous radial acceleration and the required amount and timing of impulsive delta-v correction.

Following the manner of Ref. 11, the quantities for describing the guidance scheme are $\delta(\Delta V_i)$ as the ith impulsive correction mechanization error, $\delta(\Delta\varepsilon_i)$ as the continuous acceleration correction mechanization error after the ith correction point, v_i as the radial velocity navigation error, ψ_i as the angular velocity navigation error (standard deviation), and p as the positional navigation error in radial direction, which is assumed to be constant. It is postulated that there are also correction margins, designated $\Delta\varepsilon_i$ for the ith subarc. The maximum radial distance deviation $\Delta\rho_{\max,i+1}$, known at the $(i+1)$th correction radial distance, is given by

$$E\left[\Delta\rho_{\max,i+1}^2\right] = (\alpha_i - \alpha_{i+1})^2 E\left[\delta(\Delta\varepsilon_i)^2\right] + \beta_i^2 E\left[\delta(\Delta V_i)^2\right] + \beta_i^2 E\left[v_i^2\right] + \lambda_i^2 E\left[\psi_i^2\right] + (\gamma_i - \gamma_{i+1})^2 E[p^2] \tag{13}$$

Note that the error after the $(i+1)$th correction point is not considered. This equation is written in stochastic probability form, the rss, where each quantity represents a standard deviation. When the terms on both sides of Eq. (13) are replaced by scalar variables, the largest maximum radial distance deviation is given by

$$\Delta\rho_{\max,i+1} = (\alpha_i - \alpha_{i+1})\delta(\Delta\varepsilon_i) + \beta_i\delta(\Delta V_i) + \beta_i v_i + \lambda_i\psi_i + (\gamma_i - \gamma_{i+1})p \tag{14}$$

Similar to Ref. 11, the min-max approach is adopted here to minimize the worst correction amount corresponding to the maximum deviation of the maximum radial distance. The first term of Eq. (14) is the maximum radial distance deviation as a result of unexpected fluctuations of the acceleration of the low-thrust engine. The second term denotes the contribution from mechanization error for the ith impulsive maneuver. The third and fourth term correspond to the effect of radial velocity and angular velocity navigation error for the impulsive maneuver, respectively. The fifth term represents the contribution of positional navigation error under continuous radial acceleration. Note that the contributions of positional navigation error of impulsive maneuver as well as velocity and angular positional navigation errors under continuous acceleration are of the second order or less and are neglected in this treatment. The $(i+1)$th maneuver capability $\alpha_{i+1}\Delta\varepsilon_{i+1} + \beta_{i+1}\Delta V_{i+1}$ is equal to the left-hand side of Eq. (14) plus the contribution of the $(i+1)$th navigation error, on which the $(i+1)$th maneuver is based. Therefore, the maximum radial distance deviation $\Delta\rho_{\max,i+1}$ is related to the correction strategy at the $(i+1)$th point as follows:

$$\alpha_{i+1}\Delta\varepsilon_{i+1} + \beta_{i+1}\Delta V_{i+1} = \Delta\rho_{\max,i+1} + \beta_{i+1}v_{i+1} + \lambda_{i+1}\psi_{i+1} + \gamma_{i+1}p \tag{15}$$

This equation is solved for the $(i+1)$th impulsive maneuver as

$$\Delta V_{i+1} = \frac{\beta_i}{\beta_{i+1}}\delta(\Delta V_i) + \frac{\gamma_i}{\beta_{i+1}}p + \frac{\beta_i}{\beta_{i+1}}v_i + \frac{\lambda_i}{\beta_{i+1}}\psi_i + v_{i+1} + \frac{\lambda_{i+1}}{\beta_{i+1}}\psi_{i+1} + \frac{\alpha_i - \alpha_{i+1}}{\beta_{i+1}}\delta(\Delta\varepsilon_i) - \frac{\alpha_{i+1}}{\beta_{i+1}}\Delta\varepsilon_{i+1} \tag{16}$$

The final term of Eq. (16) denotes the contribution of continuous radial acceleration control $\Delta\varepsilon_{i+1}$ to the reduction of the $(i+1)$th impulsive maneuverΔV_{i+1}. When the control capability of the continuous acceleration $\Delta\varepsilon_{i+1}$ is sufficiently large, the impulsive maneuver ΔV_{i+1} becomes zero and is not required. This yields the following Eq. (17), which provides a more realistic solution for ΔV_{i+1}. The derivation of subsequent equations, however, is based on Eq. (16).

$$\Delta V_{i+1} = \mathrm{MAX}\left[\frac{\beta_i}{\beta_{i+1}}\delta(\Delta V_i) + \frac{\gamma_i}{\beta_{i+1}}p + \frac{\beta_i}{\beta_{i+1}}v_i + \frac{\lambda_i}{\beta_{i+1}}\psi_i + v_{i+1} + \frac{\lambda_{i+1}}{\beta_{i+1}}\psi_{i+1} + \frac{\alpha_i - \alpha_{i+1}}{\beta_{i+1}}\delta(\Delta\varepsilon_i) - \frac{\alpha_{i+1}}{\beta_{i+1}}\Delta\varepsilon_{i+1}, 0\right] \tag{17}$$

The total amount of the maximum required impulsive maneuver is derived from Eq. (16) as follows:

$$\Delta V_{\text{total}} = \sum_{i=1}^{n}\Delta V_i = \sum_{i=2}^{n}\left[\frac{\beta_{i-1}}{\beta_i}\delta(\Delta V_{i-1}) + \frac{\gamma_{i-1}}{\beta_i}p + \frac{\beta_{i-1}}{\beta_i}v_{i-1} + \frac{\lambda_{i-1}}{\beta_i}\psi_{i-1} + v_i + \frac{\lambda_i}{\beta_i}\psi_i + \frac{\alpha_{i-1} - \alpha_i}{\beta_i}\delta(\Delta\varepsilon_{i-1})\right] + \Delta V_1 - \sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\Delta\varepsilon_i \tag{18}$$

The last term of the preceding equation represents the continuous radial acceleration control margin, restricted with an upper limit. Although the discussion in Ref. 11 assumes that the continuous acceleration margins are exhausted at each correction point in order to investigate the worst-case impulsive maneuver amount, the present treatment assumes that the margins are not exhausted in order to give a more feasible solution. The radial distance of the final impulsive maneuver point is determined uniquely so as to satisfy the specified allowable maximum radial distance deviation $\mathrm{MAX}(\Delta\rho_{\max})$ as follows:

$$\mathrm{MAX}(\Delta\rho_{\max}) > \alpha_n\delta(\Delta\varepsilon_n) + \beta_n\delta(\Delta V_n) + \beta_n v_n + \lambda_n\psi_n + \gamma_n p \tag{19}$$

The optimal radial distance of the impulsive maneuvers is determined by differentiating Eq. (18) with respect to ρ_i as follows:

$$\frac{\partial\Delta V_{\text{total}}}{\partial\rho_i} = \frac{\partial}{\partial\rho_i}\left[\frac{\beta_{i-1}}{\beta_i}\delta(\Delta V_{i-1}) + \frac{\beta_i}{\beta_{i+1}}\delta(\Delta V_i)\right] + \frac{\partial}{\partial\rho_i}\left(\frac{\gamma_{i-1}}{\beta_i}p + \frac{\gamma_i}{\beta_{i+1}}p\right) + \frac{\partial}{\partial\rho_i}\left(\frac{\beta_{i-1}}{\beta_i}v_{i-1} + \frac{\beta_i}{\beta_{i+1}}v_i\right) + \frac{\partial}{\partial\rho_i}\left(\frac{\lambda_{i-1}}{\beta_i}\psi_{i-1} + \frac{\lambda_i}{\beta_{i+1}}\psi_i\right) + \frac{\partial}{\partial\rho_i}\left(v_i + \frac{\lambda_i}{\beta_i}\psi_i\right) + \frac{\partial}{\partial\rho_i}\left[\frac{\alpha_{i-1} - \alpha_i}{\beta_i}\delta(\Delta\varepsilon_{i-1}) + \frac{\alpha_i - \alpha_{i+1}}{\beta_{i+1}}\delta(\Delta\varepsilon_i)\right] - \frac{\partial}{\partial\rho_i}\left(\frac{\alpha_i}{\beta_i}\Delta\varepsilon_i\right) \tag{20}$$

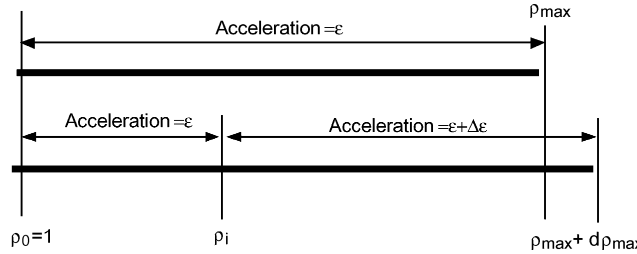

Fig. 1 Radial acceleration error and maximum distance deviation.

Finally, if $\delta(\Delta V_i)$, $\delta(\Delta\varepsilon_i)$, v_i and ψ_i are assumed to be constant, the following extended spacing rule described in the optimality condition form can be derived:

$$\frac{\partial\Delta V_{\text{total}}}{\partial\rho_i} = \left(-\frac{\beta_{i-1}}{\beta_i^2} + \frac{1}{\beta_{i+1}}\right)\frac{\partial\beta_i}{\partial\rho_i}\delta(\Delta V) + \left(-\frac{\gamma_{i-1}}{\beta_i^2}\frac{\partial\beta_i}{\partial\rho_i} + \frac{1}{\beta_{i+1}}\frac{\partial\gamma_i}{\partial\rho_i}\right)p + \left(-\frac{\beta_{i-1}}{\beta_i^2} + \frac{1}{\beta_{i+1}}\right)\frac{\partial\beta_i}{\partial\rho_i}v + \left(-\frac{\lambda_{i-1}}{\beta_i^2} + \frac{1}{\beta_{i+1}}\right)\frac{\partial\lambda_i}{\partial\rho_i}\psi + \frac{1}{\beta_i^2}\left(\frac{\partial\lambda_i}{\partial\rho_i}\beta_i - \frac{\partial\beta_i}{\partial\rho_i}\lambda_i\right)\psi + \left\{\frac{1}{\beta_i^2}\left[-\frac{\partial\alpha_i}{\partial\rho_i}\beta_i - (\alpha_{i-1} - \alpha_i)\frac{\partial\beta_i}{\partial\rho_i}\right] + \frac{1}{\beta_{i+1}}\frac{\partial\alpha_i}{\partial\rho_i}\right\}\delta(\Delta\varepsilon) - \frac{1}{\beta_i^2}\left(\frac{\partial\alpha_i}{\partial\rho_i}\beta_i - \frac{\partial\beta_i}{\partial\rho_i}\alpha_i\right)(\Delta\varepsilon_i) = 0 \tag{21}$$

Sensitivity of Radial Acceleration Error

The sensitivity of the maximum radial distance is derived here with respect to the acceleration error assuming that there exists a certain amount of acceleration error $\Delta\varepsilon$ in addition to the nominal value ε from a certain radial distance ρ_i to the maximum distance ρ_{MAX} (see Fig. 1). Then, the square of the radial velocity at a certain distance ρ between ρ_i and ρ_{MAX} can be derived from the integration of Eqs. (6–9) as follows:

$$\frac{1}{2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right)^2_{\rho=\rho} = -\frac{1}{2}\frac{1}{\rho^2} + \frac{1}{\rho} - \frac{1}{2} + \varepsilon\left(-\frac{1}{\rho_i} + 1\right) + (\varepsilon + \Delta\varepsilon)\left(-\frac{1}{\rho} + \frac{1}{\rho_i}\right) \tag{22}$$

The sensitivity of the acceleration error $\Delta\varepsilon$ with respect to the deviation of the maximum radial distance ρ_{MAX} is obtained by equating Eq. (22) with zero as follows.

$$\alpha = \frac{\partial\rho_{\max}}{\partial\Delta\varepsilon} = \frac{1 + (2\varepsilon - 1)\rho_i}{\rho_i\varepsilon(2\varepsilon - 1)^2} \tag{23}$$

The coefficient of the second term is always positive because $\rho_i < \rho_{\max} = 1/(1 - 2\varepsilon)$. The sensitivity decreases monotonically with respect to the radial distance. The partial differentiation of α with respect to the radial distance ρ_i is then as follows:

$$\frac{\partial\alpha}{\partial\rho_i} = -\frac{1}{\rho_i^2\varepsilon(2\varepsilon - 1)^2} \tag{24}$$

Sensitivity of Radial Velocity Increment

The sensitivity of the radial velocity increment (or radial velocity navigation error) at a certain radial distance ρ_i to the final maximum radial distance and its partial differentiation with respect to the radial distance ρ_i are derived from the integration of Eqs. (6–9) as follows:

$$\beta = \frac{\partial \rho_{\max}}{\partial \Delta v_i} = \frac{\sqrt{(\rho_i - 1)[1 + (2\varepsilon - 1)\rho_i]}}{\varepsilon(2\varepsilon - 1)^2 \rho_i} \tag{25}$$

$$\frac{\partial \beta}{\partial \rho_i} = \frac{1}{\rho_i^2 \varepsilon(2\varepsilon - 1)^2} \frac{1 + (\varepsilon - 1)\rho_i}{\sqrt{(\rho_i - 1)[1 + (2\varepsilon - 1)\rho_i]}} \tag{26}$$

Sensitivity of Angular Velocity Error

The sensitivity of the angular velocity navigation error at a certain radial distance ρ_i to the final maximum radial distance and its partial differentiation with respect to the radial distance ρ_i are derived from the integration of Eqs. (6–9) as follows:

$$\lambda = \frac{\partial \rho_{\max}}{\partial \psi_i} = \frac{2[(2\varepsilon - 1)\rho_i + 1]}{\varepsilon(2\varepsilon - 1)^2} \tag{27}$$

$$\frac{\partial \lambda}{\partial \rho_i} = \frac{2}{\varepsilon(2\varepsilon - 1)} \tag{28}$$

Sensitivity of Radial Distance Error

Assuming that there exists a constant radial distance navigation error Δp from a certain radial distance ρ_i to the maximum radial distance ρ_{MAX}, the sensitivity of the radial velocity increment p with respect to the maximum radial distance and its partial differentiation with respect to the radial distance ρ_i are

$$\gamma = \frac{\partial \rho_{\max}}{\partial p} = 1 - \frac{1}{(2\varepsilon - 1)^2 \rho_i^2} \tag{29}$$

$$\frac{\partial \gamma}{\partial \rho_i} = \frac{2}{\rho_i^3 (2\varepsilon - 1)^2} \tag{30}$$

Numerical Illustration of Extended Spacing Law

The extended spacing law is illustrated numerically here by minimizing the total impulsive maneuver given by Eq. (18). The minimization of the total impulsive maneuver is treated as a nonlinear programming (NLP) problem and is solved numerically by the sequential-quadratic-programming method.[12] When the number of total impulsive maneuvers is n, the performance index of the NLP is given by

$$\begin{aligned} J &= \sum_{i=2}^{n} \Delta V_i \\ &= \sum_{i=2}^{n} \left[\frac{\beta_{i-1}}{\beta_i} \delta(\Delta V_{i-1}) + \frac{\gamma_{i-1}}{\beta_i} p + \frac{\beta_{i-1}}{\beta_i} v_{i-1} + \frac{\lambda_{i-1}}{\beta_i} \psi_{i-1} \right. \\ &\quad \left. + v_i + \frac{\lambda_i}{\beta_i} \psi_i + \frac{\alpha_{i-1} - \alpha_i}{\beta_i} \delta(\Delta \varepsilon_{i-1}) \right] + \Delta V_1 - \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \Delta \varepsilon_i \end{aligned} \tag{31}$$

The parameters to be optimized are the radial distances of the impulsive maneuvers $\rho_i (i = 1 \text{ to } n - 1)$. The radial distance of the final impulsive maneuver is determined from Eq. (19). The inequality constraints are as follows:

$$\Delta V_i > 0 (i = 2 \sim n) \tag{32}$$

$$\rho_i < \rho_{i+1} (i = 1 \sim n - 1) \tag{33}$$

The assumed values of parameters are summarized in Table 1. The Earth orbit is assumed as the initial orbit, and the sun–Earth distance [i.e., 1 astronomical unit (AU)] is assumed as the unit distance. The nominal nondimensional radial acceleration ε is 0.4. Using Eq. (10), the spacecraft attains a maximum radial distance of 5 AU, corresponding to the distance of Jupiter from the sun. The nondimensional semimajor axis and the nondimensional orbital period are calculated by Eqs. (11) and (12), respectively. In Eq. (31), the contribution of the seventh term $\delta(\Delta\varepsilon)$ and the ninth and final term $\Delta\varepsilon$ are dominant if it is assumed that M2P2 technology is employed as low-thrust propulsion system.

Table 1 Parameters assumed in numerical illustration of extended spacing law

Parameters	Values
Nondimensional factor	
Sun gravity constant	0.1327124×10^{21} m^3/s^2
Sun–Earth distance (unit distance)	0.1495979×10^{12} m
Acceleration at 1 AU	0.5930083×10^{-2} m/s^2
Earth velocity (unit velocity)	29784.69 m/s
Error sources	
Radial acceleration control margin	$\Delta\varepsilon$ (parameter)
Radial acceleration error	$\delta(\Delta\varepsilon)$ (parameter)
Impulsive maneuver error	$\delta(\Delta V)$ (1 cm/s) (standard deviation)[a]
Radial velocity navigation error	v (0.1 m/s) (standard deviation)[a]
Angular velocity navigation error	ψ (10^{-8} rad/s) (standard deviation)[a]
Positional navigation error	p (100 km) (constant)[a]
Nominal trajectory	
Nominal radial acceleration at 1 AU	0.4 (ε)
Semimajor axis	3.0 AU
Apoapsis distance	5.0 AU (Jupiter distance)
Eccentricity	0.67
Orbital period	6.71 yr
Flight time = half orbital period	3.35 yr
Final impulse information	
Distance	4.999966 AU
Flight time from apoapsis	3.7574 d
Allowable maximum distance deviation	500 km

[a]Only for case 1 in Table 2. For cases 2–6, $\delta(\Delta V)$, v, ψ, and p are zero.

Table 2 lists six optimized examples, where n is the number of possible impulsive or continuous radial acceleration control maneuvers, $\delta(\Delta\varepsilon)/\varepsilon$ is the ratio of radial acceleration error (i.e., solar-wind flux variation for the M2P2 case) to the nominal radial acceleration, and $\Delta\varepsilon/\varepsilon$ is the ratio of the radial acceleration control margin (i.e., power margin generated by solar panel for the M2P2 case) to the nominal radial acceleration. The total ΔV is the total amount of impulsive maneuvers, and ΔV_1 is the initial impulsive maneuver point, representing the distance within which the spacecraft can be controlled by control of the magnitude of continuous acceleration alone. ΔV_{n-1} and ΔV_n are the $(n-1)$th and final nth impulsive delta-V, respectively. The distance is that from the sun, while the time is the flight time from the impulsive maneuver point to the final target point.

Case 1 is the only case that assumes nonzero values of $\delta(\Delta V)$, v, ψ, and p. Compared with case 2 (see Table 2), these error effects are negligible in comparison with the radial acceleration control margin $\Delta\varepsilon$ and error $\delta(\Delta\varepsilon)$. Case 2 is therefore explained thoroughly as a typical sequence, which takes the dominant terms $\Delta\varepsilon$ and $\delta(\Delta\varepsilon)$ into account and requires a small and feasible amount of impulsive maneuver. The radial acceleration error of one standard deviation (i.e., solar-wind flux variation) is 10% of the nominal value, and the radial acceleration control margin (i.e., power margin for M2P2) is 20%. These conditions require an impulsive maneuver capability of 92 m/s at most. The first impulsive maneuver is performed 21 days prior to target arrival, which means that when the flight time to the target becomes short guidance requirements cannot be achieved by control of the continuous radial acceleration magnitude alone, requiring additional impulsive correction maneuver. Table 3 shows the impulsive maneuver sequence for case 2, giving the distance from the sun, the time from the impulsive maneuver point to the target point, the magnitude of the respective impulsive maneuver, and the optimality index. The optimality index corresponds to Eq. (21) and is equal to zero when impulsive maneuvers are required. When the flight time to the target point is sufficiently long that an impulsive maneuver is not required, the optimality index is nonzero as a result of inequality conditions (32) and (33). Because of the large control

Table 2 Numerical illustration of extended spacing law

Case	n	$\delta(\Delta\varepsilon)/\varepsilon$	$\Delta\varepsilon/\varepsilon$	Total ΔV, m/s	Impulsive distance, AU	ΔV_1 time, day	ΔV_{n-1} time, day	ΔV_n time, day
1	10	0.1	0.20	96	4.9989	21.1	21.1	3.6
2	10	0.1	0.20	92	4.9989	21.4	21.4	3.8
3	10	0.1	0.10	586	4.7894	297	32.4	3.8
4	10	0.1	0.00	3538	1.8967	1057	22.5	3.8
5	20	0.1	0.00	3211	1.7211	1083	11.8	3.8
6	30	0.1	0.00	3139	1.6932	1190	9.1	3.8

Table 3 Example of continuous and impulsive maneuver sequence based on the extended spacing law (case 2 in Table 2)

No.	Distance, AU	Time, day	Delta-V, m/s	Optimality
1	1.0100000000	1212.0	0.00	——
2	2.1581196820	1017.8	0.00	294.9
3	3.4747706422	771.6	0.00	40.5
4	4.3618042226	510.8	0.00	8.7
5	4.7674825174	311.5	0.00	2.4
6	4.9200144334	183.4	0.00	0.7
7	4.9730507375	106.6	0.00	0.2
8	4.9909845177	61.6	0.00	0.0
9	4.9989197722	21.3	46.93	0.0
10	4.9999665773	3.7	45.16	0.0

Table 4 Parameters assumed in the Monte Carlo simulation of guidance scheme

Parameters	Values
Error sources	
Radial acceleration control margin $\Delta\varepsilon/\varepsilon$	20%
Radial acceleration error $\delta(\Delta\varepsilon)/\varepsilon$	10% (standard deviation)
Impulsive maneuver error $\delta(\Delta V)$	0 cm/s
Radial velocity navigation error v	0 m/s
Angular velocity navigation error ($\dot{\psi}$)	0 rad/s
Positional navigation error p	0 km
Nominal trajectory	
Nominal radial acceleration at 1 AU (ε)	0.4
Semimajor axis	3.0 AU
Apoapsis distance	5.0 AU
Eccentricity	0.67
Orbital period	6.71 yr
Guidance requirement	
Allowable max distance deviation	500 (km)(1σ)

time interval, the guidance operation might be off-line in this case with a prescheduled operation sequence.

Cases 4–6 in Table 2 assume no radial continuous acceleration control margin, and guidance is based solely on impulsive maneuvers. The total impulsive maneuver amount becomes quite large, and the optimality condition noted in Eq. (21) is satisfied at every impulsive maneuver point. As the number of possible impulsive maneuvers or opportunities for radial acceleration control (i.e., n) increases, the total amount of impulsive maneuvers becomes smaller. This means that the minimum required impulsive maneuver amount is determined by the guidance interval, which is based on the spacecraft operation scenario (i.e., tracking duration requirement for orbit determination).

Monte Carlo Simulation of the Extended Guidance Scheme

A Monte Carlo simulation was conducted for the proposed guidance scheme for radially accelerated trajectories using the parameters summarized in Table 4. For simplicity, only $\Delta\varepsilon/\varepsilon = 20\%$ and $\delta(\Delta\varepsilon)/\varepsilon = 10\%$ (standard deviation) are considered here, corresponding to case 2 in Tables 2 and 3.

Figure 2 shows nondimensional radial velocity and radial distance history as a function of flight time for a trajectory from the Earth orbit to the Jupiter distance under continuous outward radial acceleration. The initial and final radial distances are 1 and 5, respectively. Figure 3 shows an example of the predicted maximum radial distance $\rho_{\mathrm{max.predicted}}$ and the guidance command history $\Delta\varepsilon_{\mathrm{command}}$. The quantity $\rho_{\mathrm{max.predicted}}$ defined at a certain radial distance ρ_i is derived by assuming no acceleration error after the radial distance ρ_i. By setting Eq. (22) to zero, $\rho_{\mathrm{max.predicted}}$ is obtained by solving the following equation:

$$\left[1 + \rho_i^2 \left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right)^2_{\rho=\rho_i} - 2(1-\varepsilon)\rho_i\right]\rho^2_{\mathrm{max.predicted}} + 2(1-\varepsilon)\rho_i^2 \rho_{\mathrm{max.predicted}} - \rho_i^2 = 0 \tag{34}$$

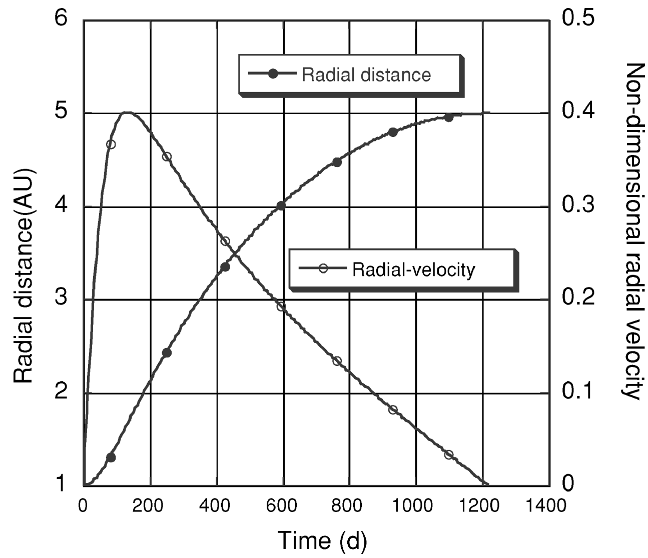

Fig. 2 Results of Monte Carlo simulation, showing radial velocity and radial distance history.

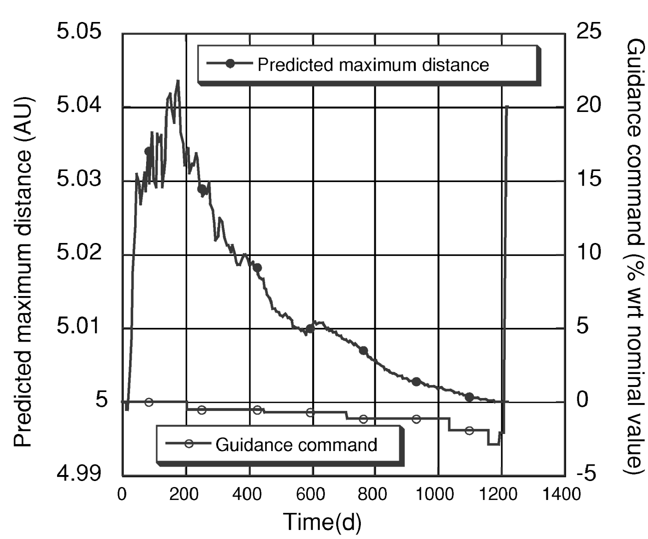

Fig. 3 Results of Monte Carlo simulation, showing predicted maximum radial distance and guidance command history.

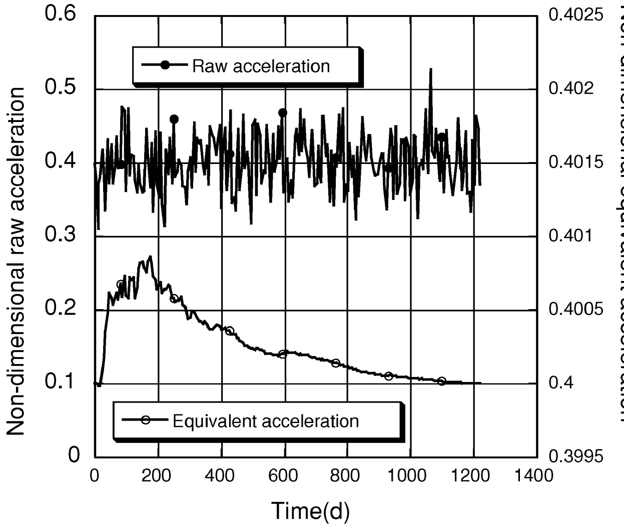

Fig. 4 Results of Monte Carlo simulation, showing raw radial acceleration and equivalent radial acceleration.

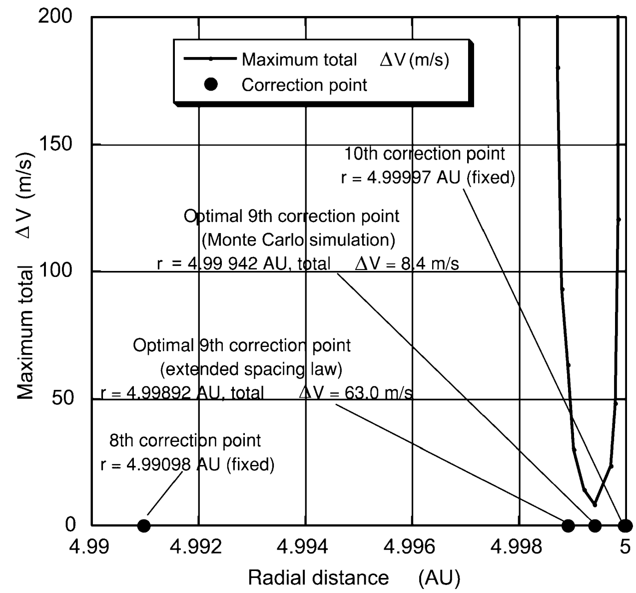

Fig. 5 Optimal correction point (analytical spacing law and Monte Carlo simulation).

When the predicted maximum radial distance is larger than the target maximum radial distance of five, the guidance command is negative in order to cancel the target maximum radial distance error. The absolute value of the guidance command increases with flight time, as the remaining flight time to the target becomes smaller. Figure 4 shows the corresponding raw radial acceleration $\varepsilon + \Delta\varepsilon$ history with 10% standard deviation error and the equivalent radial acceleration $\varepsilon_{\text{equivalent}}$ history as a function of flight time. The equivalent radial acceleration represents the total influence of the radial acceleration history, from the initial instant up to a certain distance ρ_i, and is calculated using $\rho_{\text{max.predicted}}$ and Eq. (10) as follows:

$$\varepsilon_{\text{equivalent}} = \frac{\rho_{\text{max.predicted}} - 1}{2\rho_{\text{max.predicted}}} \tag{35}$$

The guidance command $\Delta\varepsilon_{\text{command}}$ is derived simply from the sensitivity of the maximum radial distance deviation with respect to the radial acceleration increment:

$$\Delta\varepsilon_{\text{command}} = -(\rho_{\text{max.predicted}} - \rho_{\text{max.target}})/\alpha_i \tag{36}$$

where $\rho_{\text{max .target}}$ is the target maximum radial distance. When the continuous radial acceleration command is larger than the continuous radial acceleration control margin $\Delta\varepsilon$, an impulsive maneuver is required similar to Eq. (16), as follows:

$$\Delta\varepsilon_{\text{command}} = \Delta\varepsilon \tag{37}$$

$$\Delta V_i = [\text{abs}(\rho_{\text{max.predicted}} - \rho_{\text{max.target}}) - \alpha_i \Delta\varepsilon]/\beta_i \tag{38}$$

Figure 5 compares the extended spacing law results with the Monte Carlo simulation for case 2 in Table 2. The result was obtained statistically with a sufficient run number (i.e., 10,000) that provided a stable result. Correction points 1–8 and 10 are fixed, and only the radial distance of the ninth correction point was changed. The stochastic quantity [i.e., $\delta(\Delta\varepsilon)/\varepsilon$] was generated based on the Gaussian random model, and the maximum value case for the required total impulsive maneuver is plotted. The maximum radial distance deviation is 384 km, which is within the allowable deviation of 500 km. The maximum required total impulsive maneuver is 63.0 m/s, which occurs when the ninth correction point is based on the extended spacing law for a radial distance of 4.99892 AU. The overall minimum value of 8.4 m/s was achieved when the ninth correction point is located at a radial distance of 4.99942 AU. The difference in correction amount between the extended spacing law and the Monte Carlo simulation is because that the spacing law minimizes the maximum impulsive maneuver amount and provides a correction amount for the worst case. However, the difference for the optimal radial distance of the ninth correction point is only 0.0005 AU. The extended spacing law is therefore quite useful for providing an initial estimate of guidance scheme design.

Conclusions

A guidance scheme for orbital motion under continuous outward radial acceleration that is inversely proportional to the distance from the sun was investigated. The scheme treats the case of low-thrust propulsion such as minimagnetospheric plasma propulsion. The outbound trajectory was investigated, paying particular attention to the maximum attainable distance from the sun, and an extended spacing law for achieving the prescribed target maximum distance was established. The scheme was investigated numerically using a nonlinear-programming solver for an Earth–Jupiter transfer trajectory as an example, and good agreement was obtained with the Monte Carlo simulation. When the spacecraft is far from the target point, guidance is performed solely by continuous acceleration control. However, when the spacecraft approaches the target point, impulsive maneuvers are also required, as there remains insufficient time to the target point to cancel the deviation at the target by low-thrust propulsion alone. The proposed spacing law not only provides a control law for continuous radial acceleration, but also estimates the required continuous acceleration margin and amount and timing of impulsive maneuver to satisfy the terminal guidance requirement.

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